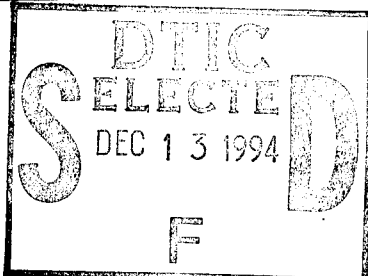


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Positioning System (GPS) and Inertial Navigation System (INS) Data.**

A Scientific Paper

Presented in Partial Fulfillment of the Requirements for

the degree Master of Science in the

Graduate School of The Ohio State University

by

Capt Jeffrey W. Haak

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## CHAPTER 1

### INTRODUCTION

The purpose of this research is to compare the effects of two filtering routines which may be used to integrate Inertial Navigation System (INS) and Global Positioning System (GPS) data to determine certain state vector elements. The two filtering routines are: 1) the ordinary Kalman Filter, and 2) a Two-Stage Least-Squares Procedure, which will be referred to as the 2-Stage Filter. Using the Kalman Filter to determine state vector elements, the vector quantities can be affected when system error is introduced into the model. Theoretically, the 2-Stage Filter is more robust, that is, it should be able to determine accurately the state vector elements despite the presence of errors (Schaffrin, 1991). This research will attempt to verify the 2-Stage Filter is, in fact, a more robust filter than the Kalman Filter.

The research used two data sets provided by Oleg Salychev and K.P. Schwarz of the University of Calgary. The GPS data are a collection of latitude and longitude measurements at a series of epochs. The INS data are a collection of latitude, longitude, velocity (in the easterly (dx) and northerly (dy) directions), and bearing measurements at a subset of the GPS epochs. By applying the model to these data sets, where 1) the difference between the INS- and GPS-observations of the easterly or northerly position of the observer, and 2) the respective velocity of the observer, are the observable elements, one can determine not only the two observable elements at a given epoch, but also two additional non-observable elements: the Direction of

the Vertical (in the nominal East-North-Up coordinate system centered at the observer), and the Drift Velocity of the observer.

Error is introduced into the model by modifying the state transition matrices,  $\Phi_x$ , and  $\Phi_y$ , the transition equation covariance matrix,  $\Theta_1$ , or the observation covariance matrix,  $\Sigma_1$ . By modifying these three matrices, the effects of the two filters on the predicted state vector elements can be observed and analyzed, and the robustness, or lack thereof, of the 2-Stage Filter can be verified. The computations were performed by a program written in C using the equations, matrices, and vectors introduced in chapters II and III.

Chapter II discusses the model used to integrate the GPS and INS data and introduces the Kalman and 2-Stage filters.

Chapter III discusses the derivation of the state transition matrices, and introduces the vectors and matrices necessary to study the filters and analyze the results.

Chapter IV presents the methods used to introduce systematic error into the model, and the results of each method.

Chapter V summarizes the results, draws the research's conclusions, and suggests areas of future research in this subject.

## CHAPTER II

### THE DYNAMIC LINEAR MODEL AND THE KALMAN AND 2-STAGE FILTERS

#### 2.1 Dynamic Linear Model

The model used to integrate the INS and GPS data is a discrete case of a Dynamic Linear Model (Schaffrin, 1990). The model observation equations are of the form:

$$y_1 = A_1 x_1 + e_1 \quad (2-1)$$

where:

$y_1$  is the (nx1) observation vector in the first epoch.

$A_1$  is the (nxm) observation coefficient matrix.

$x_1$  is the (mx1) unknown state vector of random effects at the first epoch.

$e_1$  is the (nx1) error vector of the observations whose expected value is 0, and (nxn) covariance matrix is  $\Sigma_1$ .

The state vector at the first epoch is constrained according to the state equation:

$$x_1 = \Phi_0 x_0 + u_1 \quad (2-2)$$

where:

$\Phi_0$  is the (mxm) state transition matrix from epoch 0 to epoch 1.

$x_0$  is the (mx1) unknown state vector of random effects at epoch 0.

$u_1$  is the (mx1) error vector of the states, whose expected value is 0, and (mxm) covariance matrix is  $\Theta_1$ .



Generally, it is assumed that some prior information on the initial state vector is known, hence:

$$\tilde{x}_0 = I_m x_0 + e_0^0 \quad (2-3)$$

where:

$I_m$  is the identity matrix of dimension equal to the number of elements in the state vector.

$e_0^0$  is the error vector for the initial state whose expected value is 0 and whose initial covariance matrix is  $\Sigma_0^0$ .

$e_1$ ,  $u_1$ , and  $e_0^0$  are assumed to be uncorrelated.

## **2.2 Kalman Filter**

Applying the least-squares target function:

$$\begin{aligned} \Phi(e_1, u_1, e_0^0, \lambda_1) &\equiv e_1^T \Sigma_1^{-1} e_1 + u_1^T \Theta_1^{-1} u_1 + (e_0^0)^T (\Sigma_0^0)^{-1} e_0^0 + \\ &+ 2\lambda(e_1 + A_1 u_1 - A_1 \Phi_0 e_0^0 - y_1 + A_1 \Phi_0 \tilde{x}_0) \end{aligned} \quad (2-4)$$

and solving the normal equations which satisfy the Euler-Lagrange necessary and sufficient conditions leads to the Best inhomogeneously Linear Prediction (inhom BLIP):

$$\tilde{x}_1 = \Phi_0 \tilde{x}_0 + K_1(y_1 - A_1 \Phi_0 \tilde{x}_0) \quad (2-5)$$

where  $K_1$  is the Kalman gain matrix for the first epoch and is computed:

$$K_1 = [(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)^{-1} + A_1^T \Sigma_1^{-1} A_1]^{-1} A_1^T \Sigma_1^{-1} \quad (2-6)$$

and the updated state vector covariance matrix is:

$$\Sigma_1^0 = [(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)^{-1} + A_1^T \Sigma_1^{-1} A_1]^{-1} \quad (2-7)$$

When this model is applied to the INS and GPS data, the procedure is called "Kalman Filtering" which takes the original information contained in  $\tilde{x}_0$  and the new information in the observation,  $y_1$ , together with the transition equations to yield updated information in the new vector,  $\tilde{x}_1$ .

### **2.3 2-Stage Filter**

In an effort to solve the problem of fixing cycle slips within GPS phase observations, a "two-stage least-squares procedure" is introduced (Schaffrin, 1991) which considers the observational information to be superior as long as it is available, and only in the case of missing GPS information are the INS state equations used to overcome the deficit. The 2-Stage Filter differs in that it is derived from a sequential least-squares adjustment by applying two target functions to the Dynamic Linear Model. The first target function:

$$e_1^T \Sigma_1^{-1} e_1 = (y_1 - A_1 x_1)^T \Sigma_1^{-1} (y_1 - A_1 x_1) = \min_{x_1} \quad (2-8)$$

minimizes the observational error, and only when needed, is the information from the transition equations used. Hence, the second target function, which minimizes the system error:

$$\begin{aligned} (u_1 - \Phi_0 e_0^0)^T (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)^{-1} (u_1 - \Phi_0 e_0^0) &= (x_1 - \hat{x}_1)^T (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)^{-1} (x_1 - \hat{x}_1) = \\ \min_{x_1} \{ (A_1^T \Sigma_1^{-1} A_1) x_1 &= A_1^T \Sigma_1^{-1} y_1 \} \end{aligned} \quad (2-9)$$

Since this approach only uses the transition equations when observational information is missing, it should be more resistant to the error inherent in the transition equations.

The following relations comprise the 2-Stage Filter:

$$\tilde{\tilde{x}}_1 = \hat{x}_1 + \bar{\bar{K}}_1 (y_1 - A_1 \hat{x}_1) \quad (2-10)$$

where:

$$\hat{x}_1 = \Phi_0 \tilde{x}_0 \quad (2-11)$$

$$\bar{\bar{K}}_1 = (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T [A_1 (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T]^{-1} A_1 (A_1^T \Sigma_1^{-1} A_1)^{-1} A_1^T \Sigma_1^{-1} \quad (2-12)$$

$$\bar{\bar{\Sigma}}_1^0 = (I_m - \bar{\bar{K}}_1 A_1) (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) (I_m - \bar{\bar{K}}_1 A_1)^T + \bar{\bar{K}}_1 \Sigma_1 \bar{\bar{K}}_1^T \quad (2-13)$$

Note that where there were quantities inverted in the Kalman gain matrix equation, the 2-Stage gain matrix has generalized inverses (g-inverses). The g-inverse of a matrix,  $N^-$ , is a matrix such that  $NN^-N = N$ . The need for the g-inverses arises from the rank deficiency in the observation coefficient matrix,  $A_1$ , which may cause several segments of (2-12) to become rank deficient.

Consider the first of the two g-inverses:

$$[\mathbf{A}_1(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) \mathbf{A}_1^T]^- \quad (2-12a)$$

which is a 2x2 matrix with a rank of 2. Since (2-12a) is nonsingular, the g-inverse is equivalent to the matrix inverse, that is,  $\mathbf{N}^- = \mathbf{N}^{-1}$  (Koch, 1988). The second g-inverse, which in this case is singular, is:

$$(\mathbf{A}_1^T \Sigma_1^{-1} \mathbf{A}_1)^- \quad (2-12b)$$

This is a 4x4 matrix, of rank 2, which takes the form of:

$$\begin{bmatrix} \sigma_1^{-2} & 0 & 0 & 0 \\ 0 & \sigma_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^- \quad (2-12c)$$

using  $\mathbf{A}_1$  as given in (3-15) below. Clearly, this matrix is not invertible, and the g-inverse is:

$$\begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2-12d)$$

## CHAPTER III

### VECTORS AND MATRICES INVOLVED

#### 3.1 State Transition Matrices

The state transition matrices,  $\Phi_x$  and  $\Phi_y$ , are derived from the following differential equations in spherical approximation (Salychev, 1991):

##### X-Channel

$$\delta \dot{\lambda} = \frac{1}{R \cos \varphi} \delta v_x \quad (3-1a)$$

$$\delta \dot{v}_x = -g \phi_y + B_x \quad (3-1b)$$

$$\dot{\phi}_y = \frac{\delta v_x}{R} + \varepsilon_y \quad (3-1c)$$

$$\dot{\varepsilon}_y = \omega^0 \quad (3-1d)$$

##### Y-Channel

$$\delta \dot{\varphi} = \frac{1}{R} \delta v_y \quad (3-2a)$$

$$\delta \dot{v}_y = g \phi_x + B_y \quad (3-2b)$$

$$\dot{\phi}_x = -\frac{\delta v_y}{R} + \varepsilon_x \quad (3-2c)$$

$$\dot{\varepsilon}_x = \omega^0 \quad (3-2d)$$

where:

$R$  is the mean radius of the earth.

$g$  is the gravitational constant of the earth.

$\delta \lambda$  is the observer's longitude error.

$\delta \varphi$  is the observer's latitude error

$\delta v$  is the observer's speed error in the respective direction.

$B$  is the accelerometer error.

$\phi$  is the error in the determination of the local vertical.

$\varepsilon$  is the drift rate.

$\omega^0$  is the initial drift acceleration.

The coordinate frame for the model is the East-North-Up reference frame, centered at the observer, with the x-channel corresponding with the easterly motion of the observer; the y-channel corresponding with motion in the northerly direction, and the local vertical, along the gravity vector, corresponding with the up direction. By solving these eight differential equations, the x- and y-channel transition matrices are obtained. These matrices are uncoupled and given, respectively, by (Salychev, 1991):

$$\Phi_x = \begin{bmatrix} 1 + \frac{t^2}{2R^2 \cos^2 \varphi} & \frac{t}{R \cos \varphi} & \frac{-gt^2}{2R \cos \varphi} & \frac{-gt^3}{6R \cos \varphi} \\ 0 & 1 - \frac{gt^2}{2R} & -gt & \frac{-gt^2}{2} \\ 0 & \frac{t}{R} & 1 - \frac{gt^2}{2R} & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3-3)$$

$$\Phi_y = \begin{bmatrix} 1 + \frac{t^2}{2R^2} & \frac{t}{R} & \frac{gt^2}{2R} & \frac{gt^3}{6R} \\ 0 & 1 - \frac{gt^2}{2R} & gt & \frac{gt^2}{2} \\ 0 & -\frac{t}{R} & 1 - \frac{gt^2}{2R} & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3-4)$$

The state transition equations from  $t_k$  to  $t_{k+1}$  in each channel are (Salychev, 1991):

### X-channel

$$\begin{bmatrix} \delta\lambda_{k+1} \\ \delta\nu_{k+1}^x \\ \phi_{k+1}^y \\ \varepsilon_{k+1}^y \end{bmatrix} = \Phi_x \begin{bmatrix} \delta\lambda_k \\ \delta\nu_k^x \\ \phi_k^y \\ \varepsilon_k^y \end{bmatrix} + \mathbf{u}_k^x \quad (3-5)$$

### Y-channel

$$\begin{bmatrix} \delta\varphi_{k+1} \\ \delta\nu_{k+1}^y \\ \phi_{k+1}^x \\ \varepsilon_{k+1}^x \end{bmatrix} = \Phi_y \begin{bmatrix} \delta\varphi_k \\ \delta\nu_k^y \\ \phi_k^x \\ \varepsilon_k^x \end{bmatrix} + \mathbf{u}_k^y \quad (3-6)$$

## 3.2 State Equation Covariance Matrix

The transition equation covariance matrix,  $\Theta_1$ , used was the matrix selected by Salychev and Schwarz (without explanation), and assumed to be identical for both channels with values:

$$\Theta_1 = \begin{bmatrix} 1.39 \times 10^{-4} \text{ rad}^2 & 0 & 0 & 0 \\ 0 & 1.67 \times 10^{-4} \frac{\text{m}^2}{\text{s}^2} & 6.54 \times 10^{-12} \frac{\text{rad} \cdot \text{m}}{\text{s}} & 0 \\ 0 & 6.54 \times 10^{-12} \frac{\text{rad} \cdot \text{m}}{\text{s}} & 3.42 \times 10^{-12} \text{ rad}^2 & 0 \\ 0 & 0 & 0 & 9.4 \times 10^{-12} \frac{\text{rad}^2}{\text{s}^2} \end{bmatrix} \quad (3-7)$$

### 3.3 Initial State Vector

The initial value for the state vector,  $x_0$  was assumed for both channels to be:

$$x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3-8)$$

### 3.4 State Vector Initial Covariance Matrix

The initial state vector covariance matrix,  $\Sigma_0^0$ , courtesy of Salychev and Schwarz, was also assumed to be identical for both channels with values:

$$\Sigma_0^0 = \begin{bmatrix} 1.4 \times 10^{-4} \text{ rad}^2 & 0 & 0 & 0 \\ 0 & 1.0 \times 10^{-4} \frac{\text{m}^2}{\text{s}^2} & 0 & 0 \\ 0 & 0 & 5.0 \times 10^{-17} \text{ rad}^2 & 0 \\ 0 & 0 & 0 & 5.0 \times 10^{-17} \frac{\text{rad}^2}{\text{s}^2} \end{bmatrix} \quad (3-9)$$

### 3.5 Observation Equations

In each channel, there are two observation equations (Salychev, 1991):

#### X-Channel

$$\delta\lambda = \lambda_{GPS} - \lambda_{ins} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\lambda_k \\ \delta\nu_k^x \\ \phi_k^y \\ \varepsilon_k^y \end{bmatrix} + e_{\delta\lambda} \quad (3-10)$$



$$\delta v_x = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \lambda_k \\ \delta v_k^x \\ \phi_k^y \\ \varepsilon_k^y \end{bmatrix} + e_{\delta v_x} \quad (3-11)$$

### Y-Channel

$$\delta \varphi = \varphi_{GPS} - \varphi_{ins} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \varphi_k \\ \delta v_k^y \\ \phi_k^x \\ \varepsilon_k^x \end{bmatrix} + e_{\delta \varphi} \quad (3-12)$$

$$\delta v_y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \varphi_k \\ \delta v_k^y \\ \phi_k^x \\ \varepsilon_k^x \end{bmatrix} + e_{\delta v_y} \quad (3-13)$$

which, using matrix notation, takes the form:

$$\mathbf{y}_1 = \mathbf{A}_1 \mathbf{x}_1 + \mathbf{e}_1 \quad (3-14)$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (3-15)$$

### **3.6 Observation Covariance Matrix**

The covariance matrix of the observation equations,  $\Sigma_1$ , courtesy of Salychev and Schwarz, was, again, assumed to be identical for both channels with values:

$$\Sigma_1 = \begin{bmatrix} 2.0 \times 10^{-6} \text{ rad}^2 & 0 \\ 0 & 2.25 \times 10^{-6} \frac{m^2}{s^2} \end{bmatrix} \quad (3-16)$$

## CHAPTER IV

### RESULTS AND ANALYSES

Error was introduced into the model in one of three ways: 1) modifying the state transition covariance matrix,  $\Theta_1$ , 2) modifying the state transition matrices,  $\Phi_x$ , and  $\Phi_y$ , and 3) modifying the observation covariance matrix,  $\Sigma_1$ .

#### 4.1 Increasing the Magnitude of $\Theta_1$ .

**Hypothesis:** When system error is introduced by increasing the magnitude of the state transition covariance matrix,  $\Theta_1$ , state vectors computed by the filter which is more sensitive to this type of error display larger element magnitude changes than the state vectors of the less sensitive filter. The 2-Stage Filter, due to the fact that it only uses the information from the transition equations in the absence of observations, is expected to be the less sensitive filter.

**Procedure:** A modified  $\Theta_1$  with diagonal elements increased to 4000 (which is sufficiently large to approximate infinity) and off-diagonal elements reduced to 0 was introduced. The results of both filters were observed to determine the sensitivities of each of them to the introduced systematic error.

**Results:** When the magnitude of  $\Theta_1$  is increased to approximate infinity, both filters yielded identical state vectors for a given set of state transition matrices. The first two state vector elements reproduced the observations, with the third and fourth elements equalling values greatly distorted from the unmodified computed values. **Table 1** displays a portion of the results for the case where the transition matrices were modified. The first two state vector elements from the 2-Stage Filter were insignificantly affected by the change in  $\Theta_1$ , however, the same cannot be said for the Kalman Filter. **Table 2** shows the changes in the first two state vector elements of the Kalman Filter at selected epochs. Two lines are included for each epoch: the first line includes the observed quantities and the unmodified first two state vector elements; the second includes the first state vector elements computed when  $\Theta_1$  becomes large and approaches  $\infty$ . The epochs of interest correspond with the initial epoch after a cycle slip.

**Analysis:** Since a number of the first and second elements of the Kalman Filter state vectors were affected by the introduced system error (**Table 2**), and the respective 2-Stage Filter state vector elements (not shown) were unaffected, it appears the 2-Stage Filter is less sensitive to this type of system error than the Kalman Filter. The reason for this insensitivity can be seen in the gain matrices.

As  $\Theta_1$  approaches  $\infty$ , the Kalman gain matrix,  $K_1$ , is transformed as follows:

$$K_1 = [(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)^{-1} + A_1^T \Sigma_1^{-1} A_1]^{-1} A_1^T \Sigma_1^{-1} \quad (4-1)$$

since  $\Theta_1$  dominates the  $\Phi_0 \Sigma_0^0 \Phi_0^T$  term:

$$K_1 = [(\Theta_1)^{-1} + A_1^T \Sigma_1^{-1} A_1]^{-1} A_1^T \Sigma_1^{-1} \quad (4-1a)$$

since the inverse of  $\infty$  is 0:

$$\mathbf{K}_1 = [\mathbf{A}_1^T \Sigma_1^{-1} \mathbf{A}_1]^{-1} \mathbf{A}_1^T \Sigma_1^{-1}. \quad (4-1b)$$

Noting that  $\mathbf{A}_1$  is a matrix made up of  $\mathbf{I}_2$  and the 2x2 zero matrix, and  $\Sigma_1$  is diagonal,  $\mathbf{K}_1$  reduces to:

$$\mathbf{K}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (4-1c)$$

at the initial epoch. At later epochs, however, the lower block elements will become much larger. To see why this is so, one needs to look to the equation to update the state vector covariance matrix:

$$\Sigma_1^0 = [(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)^{-1} + \mathbf{A}_1^T \Sigma_1^{-1} \mathbf{A}_1]^{-1}. \quad (4-2)$$

When  $\Theta_1$  is large, the third and fourth diagonal elements of the  $\Sigma_1^0$  also become large which causes the lower 2x2 block in the gain matrix to assume values other than 0. Since the state equation covariance matrix represents the error in the prior information, when the third and fourth diagonal elements become large, and approach  $\infty$ , this implies that the observer knows nothing concerning the third and fourth elements of the state vector. Consequently, the values which are computed through the transition equations with such error present are, essentially, arbitrary, and the magnitude of the change in either filter's state vector is irrelevant. Rewriting the state transition equation:

$$\tilde{\mathbf{x}}_I = \begin{bmatrix} \hat{x}_1^1 \\ \hat{x}_2^1 \\ \hat{x}_3^1 \\ \hat{x}_4^1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix} \left( \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1^1 \\ \hat{x}_2^1 \\ \hat{x}_3^1 \\ \hat{x}_4^1 \end{bmatrix} \right) \quad (4-3)$$

which reduces to:

$$\tilde{\mathbf{x}}_I = \begin{bmatrix} \hat{x}_1^1 \\ \hat{x}_2^1 \\ \hat{x}_3^1 \\ \hat{x}_4^1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix} \begin{pmatrix} y_1' - \hat{x}_1^1 \\ y_2' - \hat{x}_2^1 \end{pmatrix} \quad (4-3a)$$

it becomes apparent that the coefficient matrix filters out the third and fourth elements of the state vector. Therefore, the  $\hat{x}_3^1$  and  $\hat{x}_4^1$  elements are eliminated, however, the  $\hat{x}_1^1$  and  $\hat{x}_2^1$  elements remain and are, generally, non-zero.

The 2-Stage Filter gain matrix is similarly affected.

$$\bar{\bar{\mathbf{K}}}_1 = (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) \mathbf{A}_1^T [\mathbf{A}_1 (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) \mathbf{A}_1^T]^{-1} \mathbf{A}_1 (\mathbf{A}_1^T \Sigma_1^{-1} \mathbf{A}_1)^{-1} \mathbf{A}_1^T \Sigma_1^{-1} \quad (4-4)$$

As  $\Theta_1$  becomes large, it dominates the  $\Phi_0 \Sigma_0^0 \Phi_0^T$  term and the gain matrix,  $\bar{\bar{\mathbf{K}}}_1$ , is reduced at the initial epoch to:

$$\bar{\bar{\mathbf{K}}}_1 = \Theta_1 \mathbf{A}_1^T [\mathbf{A}_1 \Theta_1 \mathbf{A}_1^T]^{-1} \mathbf{A}_1 (\mathbf{A}_1^T \Sigma_1^{-1} \mathbf{A}_1)^{-1} \mathbf{A}_1^T \Sigma_1^{-1} \quad (4-4a)$$

The segment following the first g-inverse reduces to  $\mathbf{I}_2$ , and  $[\mathbf{A}_1 \Theta_1 \mathbf{A}_1^T]$  is a 2x2 matrix with large diagonal elements, which reduces to:

$$\bar{\bar{\mathbf{K}}}_1 = \begin{bmatrix} \theta_{11} & 0 & 0 & 0 \\ 0 & \theta_{22} & 0 & 0 \\ 0 & 0 & \theta_{33} & 0 \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{11} & 0 \\ 0 & \theta_{22} \end{bmatrix}^{-1} \quad (4-4b)$$

which equals:

$$\bar{\bar{\mathbf{K}}}_1 = \begin{bmatrix} \theta_{11} & 0 \\ 0 & \theta_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{11}^{-1} & 0 \\ 0 & \theta_{22}^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (4-4c)$$

Furthermore, in the limit,  $\bar{\bar{\Sigma}}_1^1 = \Sigma_0^1$ . This equivalence is not readily apparent from equations (2-7) and (2-13), however, an equivalent form of (2-7) is:

$$\Sigma_1^0 = (\mathbf{I}_m - \mathbf{K}_1 \mathbf{A}_1)(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)(\mathbf{I}_m - \mathbf{K}_1 \mathbf{A}_1)^T + \mathbf{K}_1 \Sigma_1 \mathbf{K}^T \quad (4-5)$$

which closely resembles the 2-Stage expression:

$$\bar{\bar{\Sigma}}_1^0 = (\mathbf{I}_m - \bar{\bar{\mathbf{K}}}_1 \mathbf{A}_1)(\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T)(\mathbf{I}_m - \bar{\bar{\mathbf{K}}}_1 \mathbf{A}_1)^T + \bar{\bar{\mathbf{K}}}_1 \Sigma_1 \bar{\bar{\mathbf{K}}}^T \quad (4-6)$$

Therefore, at each epoch, the  $\hat{x}_1^1$  and  $\hat{x}_2^1$  elements are eliminated, leaving the observations,  $y_1^1$  and  $y_2^1$ , and  $\hat{x}_3^1$  and  $\hat{x}_4^1$  as the state vector elements from each filter.

This case illustrates that the 2-Stage Filter is insensitive to error introduced through the state transition covariance matrix for those state vector elements which have been

observed, though it is not more robust than the Kalman Filter since it computes identical state vectors . It also illustrates that both filters become unstable for those state vector elements which are not observed.

Another way of looking at this situation is that when we increase the magnitude of the state transition covariance matrix, we are telling the model we do not have a good set of equations to transition from one state to another. Consequently, the only useful state vector elements which can be expected are those which we can observe, which both filters provide.



**Table 1**  
**State Vector Comparison**

$$\Theta_1 \approx \infty$$

**Kalman Filter**

Observed Quantities			Updated State Vector			
Epoch	dLambda	dVelocity	dLambda	dVelocity	Attitude Angle	Drift
sec	rad	m/sec	rad	m/sec	rad	rad/sec
2	-2.461E-5	7.877E-2	-2.461E-5	7.877E-2	6.181E-16	-1.929E-20
4	-2.466E-5	7.917E-2	-2.466E-5	7.917E-2	-3.072E-5	-1.025E-5
6	-2.471E-5	7.978E-2	-2.471E-5	7.978E-2	-3.592E-5	-5.163E-6
8	-2.476E-5	8.038E-2	-2.476E-5	8.038E-2	-3.078E-5	-1.923E-8
14	-2.490E-5	8.219E-2	-2.490E-5	8.219E-2	2.535E-5	3.569E-5
16	-2.496E-5	8.360E-2	-2.496E-5	8.360E-2	-1.023E-4	-3.060E-5
18	-2.501E-5	8.360E-2	-2.501E-5	8.360E-2	3.563E-5	3.571E-5
20	-2.506E-5	8.360E-2	-2.506E-5	8.360E-2	1.587E-7	9.929E-8
22	-2.510E-5	8.520E-2	-2.510E-5	8.520E-2	-1.229E-4	-4.098E-5
24	-2.515E-5	8.520E-2	-2.515E-5	8.520E-2	4.073E-5	4.082E-5
26	-2.520E-5	8.520E-2	-2.520E-5	8.520E-2	1.853E-7	1.190E-7
150	-2.802E-5	1.254E-1	-2.802E-5	1.254E-1	-6.616E-5	-5.470E-7
156	-2.817E-5	1.271E-1	-2.817E-5	1.271E-1	-5.902E-5	-7.751E-6
158	-2.821E-5	1.271E-1	-2.821E-5	1.271E-1	2.554E-5	2.556E-5
3672	5.612E-4	4.883E-1	5.612E-4	4.883E-1	1.656E-4	4.779E-5

**2-Stage Filter**

Observed Quantities			Updated State Vector			
Epoch	dLambda	dVelocity	dLambda	dVelocity	Attitude Angle	Drift
sec	rad	m/sec	rad	m/sec	rad	rad/sec
2	-2.461E-5	7.877E-2	-2.461E-5	7.877E-2	6.181E-16	-1.929E-20
4	-2.466E-5	7.917E-2	-2.466E-5	7.917E-2	-3.072E-5	-1.025E-5
6	-2.471E-5	7.978E-2	-2.471E-5	7.978E-2	-3.592E-5	-5.163E-6
8	-2.476E-5	8.038E-2	-2.476E-5	8.038E-2	-3.078E-5	-1.923E-8
14	-2.490E-5	8.219E-2	-2.490E-5	8.219E-2	2.535E-5	3.569E-5
16	-2.496E-5	8.360E-2	-2.496E-5	8.360E-2	-1.023E-4	-3.060E-5
18	-2.501E-5	8.360E-2	-2.501E-5	8.360E-2	3.563E-5	3.571E-5
20	-2.506E-5	8.360E-2	-2.506E-5	8.360E-2	1.587E-7	9.929E-8
22	-2.510E-5	8.520E-2	-2.510E-5	8.520E-2	-1.229E-4	-4.098E-5
24	-2.515E-5	8.520E-2	-2.515E-5	8.520E-2	4.073E-5	4.082E-5
26	-2.520E-5	8.520E-2	-2.520E-5	8.520E-2	1.853E-7	1.190E-7
150	-2.802E-5	1.254E-1	-2.802E-5	1.254E-1	-6.616E-5	-5.470E-7
152	-2.808E-5	1.254E-1	-2.808E-5	1.254E-1	1.946E-5	1.949E-5
158	-2.821E-5	1.271E-1	-2.821E-5	1.271E-1	2.554E-5	2.556E-5
3672	5.612E-4	4.883E-1	5.612E-4	4.883E-1	1.656E-4	4.779E-5

**Table 2**  
**Kalman Filter State Vector Observed Element Changes**  
 $\Theta_1$  versus  $\Theta_\infty$

	Observed Quantities			Updated State Vector Elements	
	Epoch	dLambda	dVelocity	dLambda	dVelocity
	sec	rad	m/sec	rad	m/sec
$\Theta_1$	2	-2.461E-5	7.877E-2	-2.443E-5	7.811E-2
$\Theta_\infty$	2			-2.461E-5	7.877E-2
$\Theta_1$	4	-2.466E-5	7.917E-2	-2.466E-5	7.916E-2
$\Theta_\infty$	4			-2.466E-5	7.917E-2
$\Theta_1$	6	-2.471E-5	7.978E-2	-2.471E-5	7.977E-2
$\Theta_\infty$	6			-2.471E-5	7.978E-2
$\Theta_1$	150	-2.802E-5	1.254E-1	-2.794E-5	1.254E-1
$\Theta_\infty$	150			-2.802E-5	1.254E-1
$\Theta_1$	332	-3.171E-5	1.752E-1	-3.158E-5	1.752E-1
$\Theta_\infty$	332			-3.171E-5	1.752E-1
$\Theta_1$	473	-3.362E-5	2.194E-1	-3.352E-5	2.194E-1
$\Theta_\infty$	473			-3.362E-5	2.194E-1
$\Theta_1$	656	-3.440E-5	2.781E-1	-3.427E-5	2.781E-1
$\Theta_\infty$	656			-3.440E-5	2.781E-1
$\Theta_1$	806	-3.375E-5	3.529E-1	-3.363E-5	3.529E-1
$\Theta_\infty$	806			-3.375E-5	3.529E-1
$\Theta_1$	810	-3.374E-5	3.368E-1	-3.374E-5	3.370E-1
$\Theta_\infty$	810			-3.374E-5	3.368E-1
$\Theta_1$	1002	-3.151E-5	4.437E-1	-3.131E-5	4.437E-1
$\Theta_\infty$	1002			-3.151E-5	4.437E-1
$\Theta_1$	3137	4.564E-4	8.882E-1	4.561E-4	8.882E-1
$\Theta_\infty$	3137			4.564E-4	8.882E-1

#### **4.2 Modifying Elements of the Transition Matrices.**

**Hypothesis:** If, when system error is introduced through modifying the transition matrices, the elements of the state vector computed by one of the filters are changed more substantially than the other, it would be evidence that the filter is more sensitive to the error in the model. For a given pair of modified transition matrices, the 2-Stage Filter is expected to calculate state vectors whose elements are less significantly changed than those of the Kalman Filter because of its acceptance of the observations as superior to the transition equations.

**Procedure:** The filter which is less susceptible to system error would display smaller magnitude changes in its state vector elements than the other. The change in the state vector can be quantified by computing the Euclidean vector norm:

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \quad (4-7)$$

at each epoch. By modifying  $\Phi_x$  and  $\Phi_y$  and observing the changes in the state vector norms for each filter, evidence should become apparent concerning which filter is more sensitive to system error in the model. System error is introduced by modifying the transition matrices in one of three ways: 1) Using a value of 15 m/s<sup>2</sup> for the gravitational acceleration of the earth; 2) Using a value of 2,000,000 m for the mean radius of the earth; and 3) Multiplying the unmodified transition matrices by the constant value 2.

**Results:** In all three cases, the observed state vector elements were unaffected by the system error, and the modified state vector norms for both filters changed in similar manners over the observation set. **Figures 1, 3, and 5** depict the vector norm percentage changes for each case.

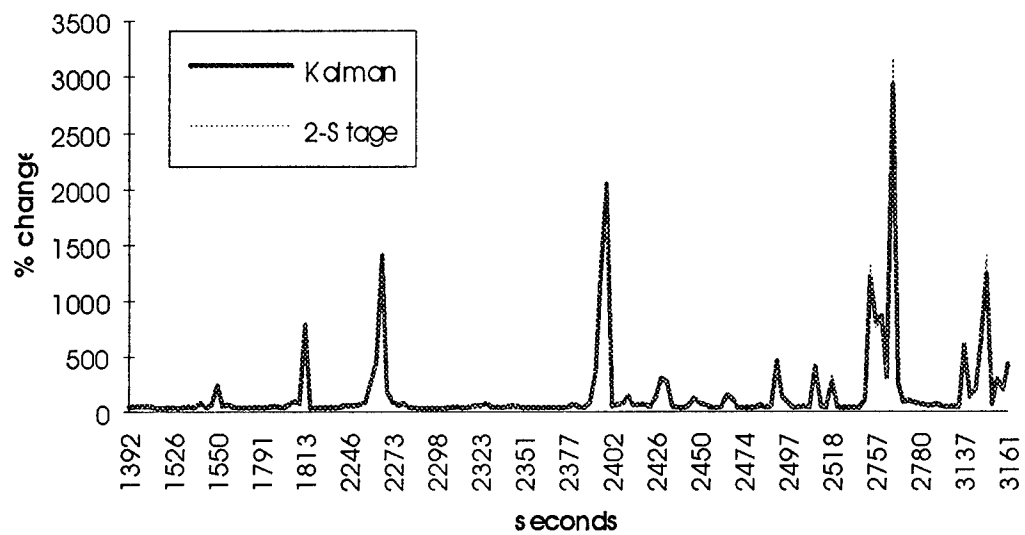
**Analysis:** Due to the large differences in magnitude between the observable and non-observable state vector elements, simply computing the percentage difference between the norms of the modified and unmodified state vectors was inconclusive. The much larger observed elements were unaffected by the system error, and caused the vector norms to be unchanged. However, when a third vector comprised of elements which were the percentage differences between the respective state vector elements, large fluctuations became apparent in the norm percentage changes for both filters. Unfortunately, the state vector norms changed in such a similar fashion that the difference between the two curves in **Figures 1, 3, and 5** is indistinguishable. To determine if a statistically significant difference exists between the results of each filter, the difference between the Kalman and 2-Stage state vector norm percentage changes were computed at each epoch (**Figures 2, 4, and 6**). For a normal distribution, 68.3 percent of the area lies within one standard deviation of the distribution mean. Therefore, for a mean which is close to 0, if a higher percentage of the differences lie within one standard deviation of the mean, it can be interpreted to mean there is no statistical difference between the results of the two filters. Conversely, if the number of differences within one standard deviation of the mean is lower than 68.3 percent, or the mean is a value other than 0, it can be interpreted as statistically significant differences between the results of the two filters.

**Case 1:** When system error is introduced by adjusting the value for the gravitational constant to  $15 \text{ m/s}^2$ , from  $9.8 \text{ m/s}^2$ , the state vector norms from both filters are similar, as shown in **Figure 1**. However, when the differences between the Kalman and 2-Stage filters are plotted (**Figure 2**), it is apparent that the mean is close to 0, and very few of the points lie outside the region bounded by one standard deviation either side of the mean (indicated by the straight lines). Of the 172 points displayed, only 11 (6%) lie outside the region (which is far fewer than what would be expected for a normal distribution) suggesting that there is no statistical difference between the results of the two filters in this case.

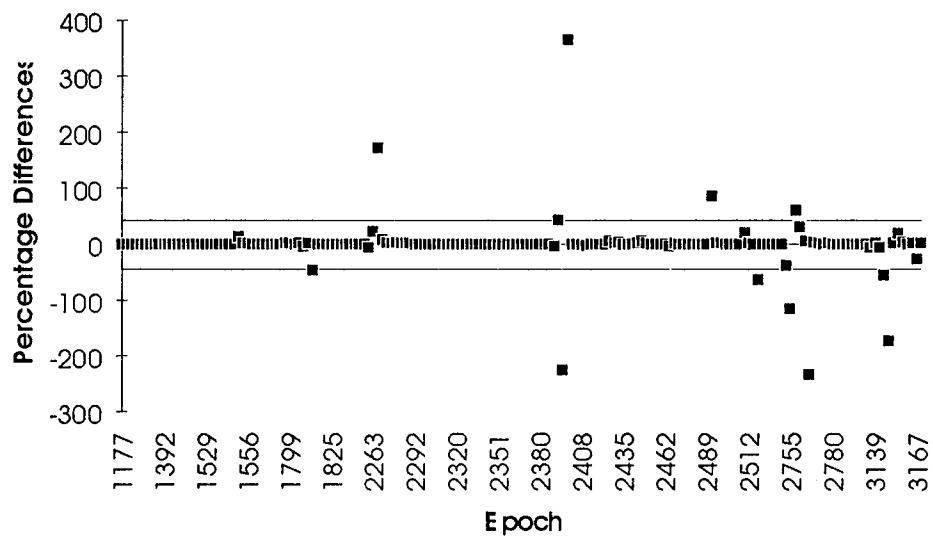
**Case 2:** When system error is introduced by adjusting the mean radius of the earth to be 2,000,000 m, from 6,701,000 m, the results are similar to Case 1. The state vector norms are similar, as seen in **Figure 3**, and the percentage differences of the vector norms lie predominantly with one standard deviation around 0. In this case, the amount of error introduced was greater, which can be seen in the larger magnitudes of the differences, however, both filters were affected in a similar fashion, and no statistically significant difference is apparent between the results.

**Case 3:** When system error is introduced by multiplying the unmodified transition matrix by 2, the results, again, reflect a lack of any statistically significant difference between the Kalman and 2-Stage filters. **Figure 5** clearly shows the state vector norms to be indistinguishable, and **Figure 6** shows few differences, six, to lie beyond the region bounded by one standard deviation. The magnitudes of the differences are significantly larger due to the larger error introduced, but the differences between the two filters appears to be statistically insignificant.

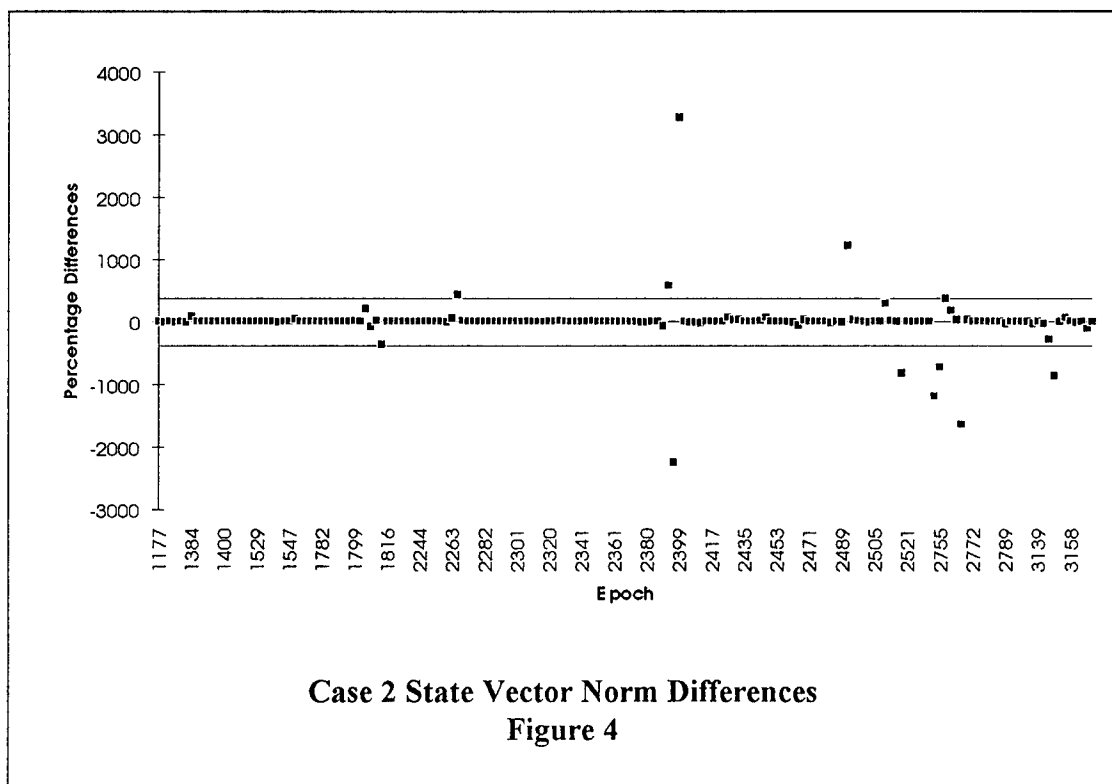
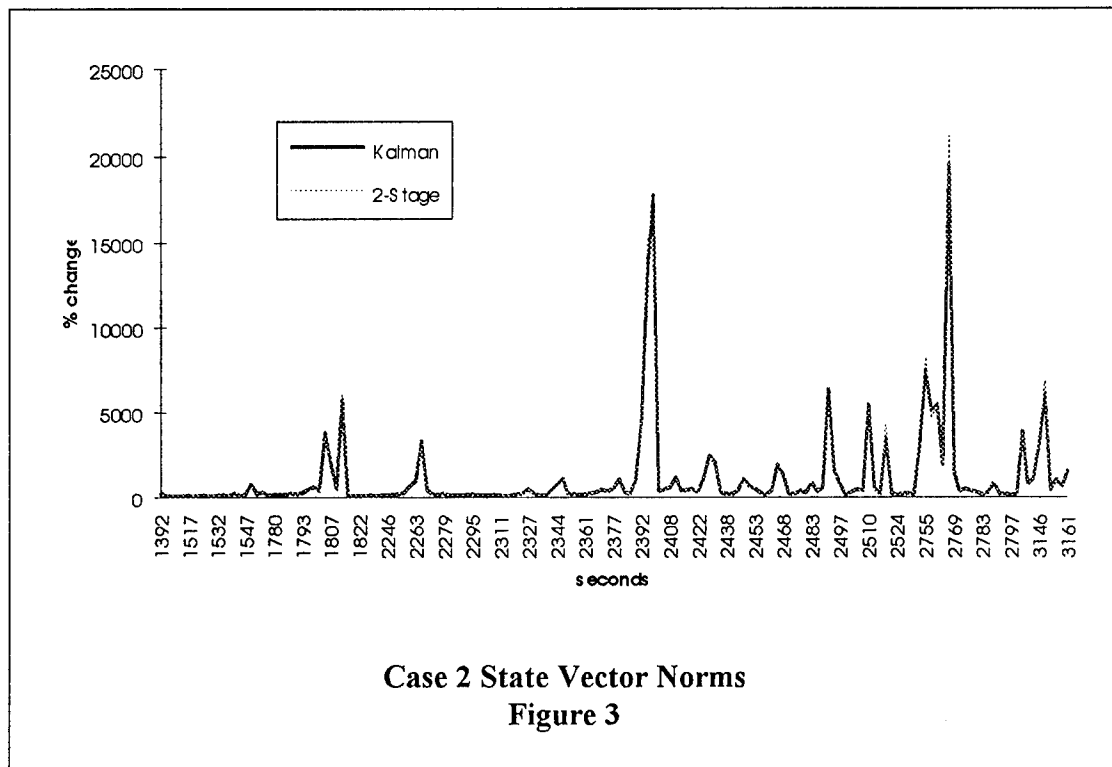
As was the case when we introduced system error through the state transition covariance matrix, when the transition matrices are distorted, we are conceding we do not have a good set of equations to transition from one state to the next. Consequently, one cannot expect any stable state vector elements other than what can be observed, which both filters, again, displayed.



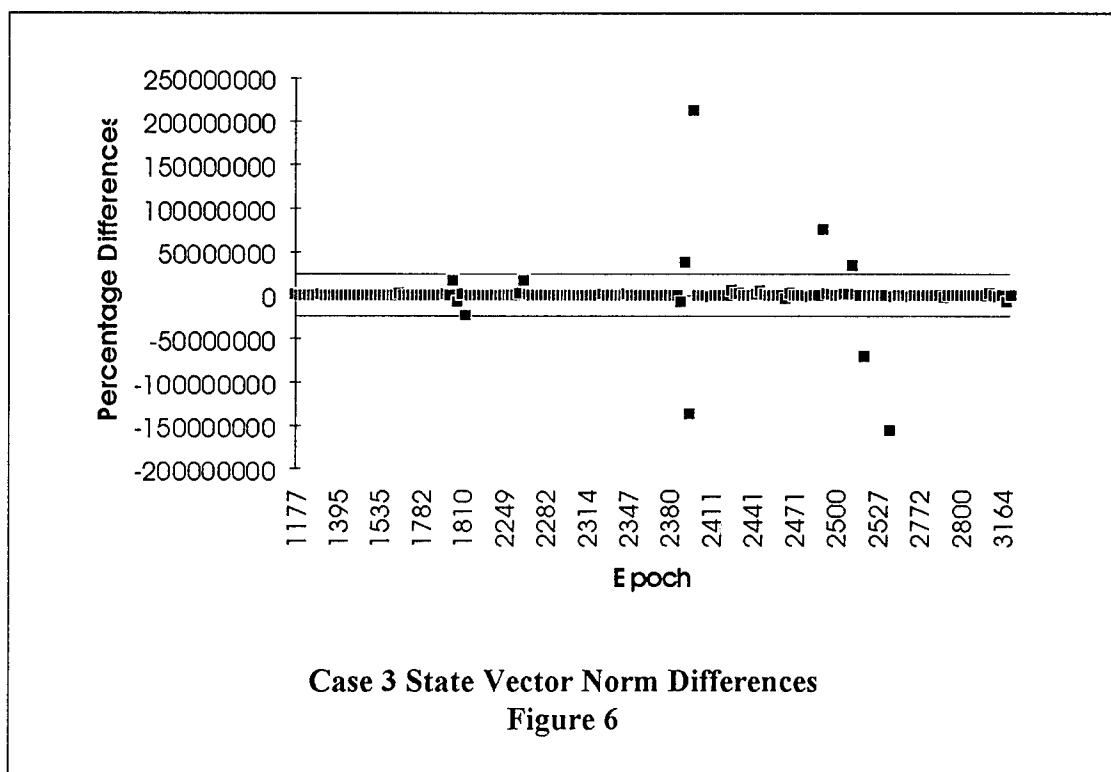
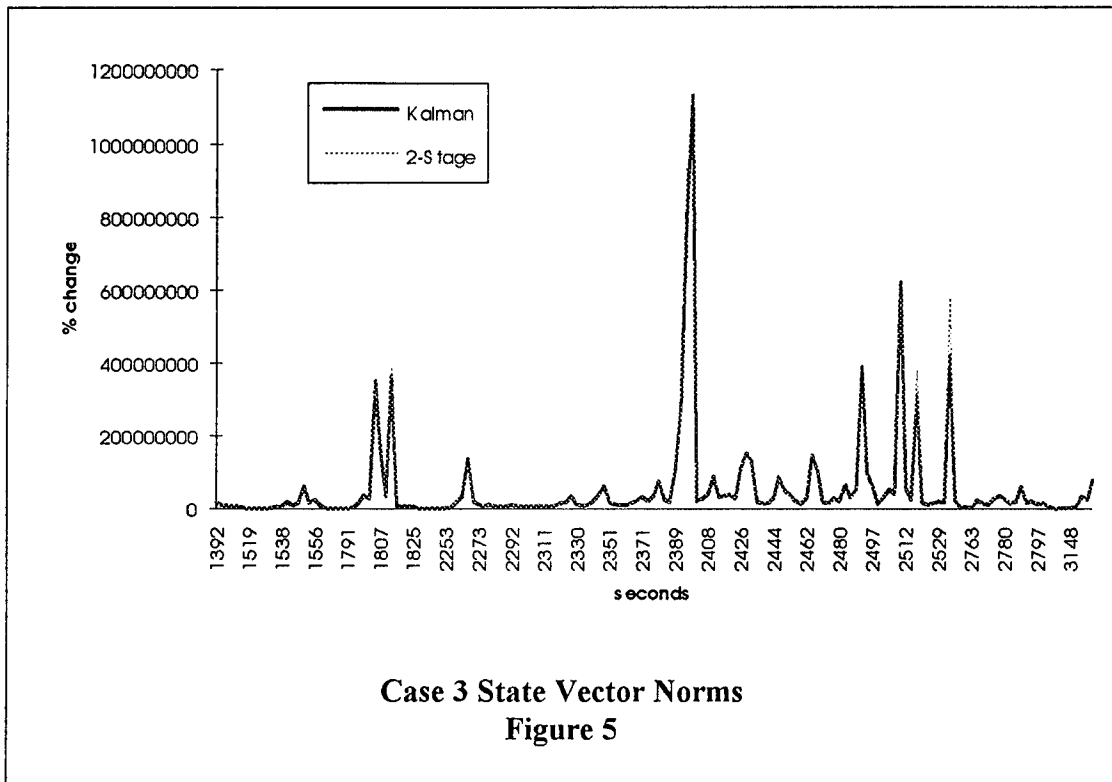
Case 1 State Vector Norms  
Figure 1



Case 1 State Vector Norm Differences  
Figure 2







### **4.3 Severely Distorting the Transition Matrices.**

**Hypothesis:** The filter which is more robust will continue to compute the two observed quantities even when systematic error is introduced which so distorts the transition matrices such that the less robust filter will yield state vectors whose first two elements fail to reproduce the two observed values.

**Procedure:** The transition matrices were modified by multiplying the unmodified matrices by a 500. The results were observed to determine the effects on the first two elements of the state vectors of both filters.

**Results:** The results for a number of the epochs are included in **Table 3**. The first two elements of the 2-Stage Filter-computed state vectors reproduce the observed quantities to the same precision, whereas, at the same epochs, the Kalman Filter-computed state vector show the first element to change in the second and third decimal place, and the second element to, on occasion, change as well. At epoch 150, there is a more severe distortion in the first element for the Kalman Filter which appears throughout the data set at the first epoch following a cycle slip, however, similar distortions also are present after cycle slips in the 2-Stage Filter results.

**Analysis:** **Table 3** clearly shows the 2-Stage Filter to be relatively unaffected by this type of systematic error, except for the periods immediately following cycle slips. A second modified matrix was used in which the unmodified transition matrices were multiplied by 50,000. Though the introduction of this much error might appear ridiculous, the results are interesting. **Table 4** shows the Kalman Filter state vector

elements to be surprisingly stable, with results meeting, and often exceeding, the precision in **Table 3**. The 2-Stage results were much less stable. This apparent instability, however, was not due to the breaking down of the 2-Stage Filter, but rather, caused by precision rounding by the operating system. At times, the 2-Stage Filter was reproducing the observed precision, however, over time, rounding errors eventually cause the filter to return "NaN" (for values which are not a number) in the computation of the gain matrix elements, resulting in useless results. From this perspective, it appears that the Kalman Filter is, in a manner of speaking, less sensitive to this type of error in that, even though it may lose some of its state vector element precision, it does not appear to be as severely affected by system round-off as the 2-Stage Filter.

Unlike the previous situations, the observed elements did not pass through the Kalman filter without distortion, albeit relatively minor distortion given the magnitude of the system error. Unfortunately, the 2-Stage filter was not unaffected by the error, which suggests an area for possible future consideration.

**Table 3**  
**State Vector Observed Element Distortion**

$$\Phi_{\text{mod}} = 500\Phi$$

**Kalman Filter**  
**Observed Quantities      State Vector Elements**

	Epoch	dLambda	dVelocity	dLambda	dVelocity
	sec	rad	m/sec	rad	m/sec
$\Phi$	2	-2.461E-5	7.877E-2	-2.443E-5	7.811E-2
$\Phi_{\text{mod}}$	2			-2.461E-5	7.877E-2
$\Phi$	4	-2.466E-5	7.917E-2	-2.466E-5	7.916E-2
$\Phi_{\text{mod}}$	4			-2.471E-5	7.933E-2
$\Phi$	150	-2.802E-5	1.254E-1	-2.794E-5	1.254E-1
$\Phi_{\text{mod}}$	150			-5.414E-5	1.254E-1
$\Phi$	152	-2.808E-5	1.254E-1	-2.808E-5	1.254E-1
$\Phi_{\text{mod}}$	152			-2.814E-5	1.254E-1
$\Phi$	2500	2.473E-4	1.050E+0	2.473E-4	1.050E+0
$\Phi_{\text{mod}}$	2500			2.477E-4	1.050E+0
$\Phi$	2502	2.482E-4	1.049E+0	2.482E-4	1.049E+0
$\Phi_{\text{mod}}$	2502			2.487E-4	1.049E+0
$\Phi$	3661	5.601E-4	4.965E-1	5.601E-4	4.965E-1
$\Phi_{\text{mod}}$	3661			5.612E-4	4.965E-1

**2-Stage Filter**  
**Observed Quantities      State Vector Elements**

	Epoch	dLambda	dVelocity	dLambda	dVelocity
	sec	rad	m/sec	rad	m/sec
$\Phi$	2	-2.461E-5	7.877E-2	-2.461E-5	7.877E-2
$\Phi_{\text{mod}}$	2			-2.461E-5	7.877E-2
$\Phi$	4	-2.466E-5	7.917E-2	-2.466E-5	7.917E-2
$\Phi_{\text{mod}}$	4			-2.466E-5	7.917E-2
$\Phi$	150	-2.802E-5	1.254E-1	-2.802E-5	1.254E-1
$\Phi_{\text{mod}}$	150			-2.802E-5	1.254E-1
$\Phi$	152	-2.808E-5	1.254E-1	-2.808E-5	1.254E-1
$\Phi_{\text{mod}}$	152			-2.808E-5	1.254E-1
$\Phi$	2500	2.473E-4	1.050E+0	2.473E-4	1.050E+0
$\Phi_{\text{mod}}$	2500			2.473E-4	1.050E+0
$\Phi$	2502	2.482E-4	1.049E+0	2.482E-4	1.049E+0
$\Phi_{\text{mod}}$	2502			2.482E-4	1.049E+0
$\Phi$	3661	5.601E-4	4.965E-1	5.601E-4	4.965E-1
$\Phi_{\text{mod}}$	3661			5.601E-4	4.965E-1

**Table 4**  
**State Vector Observed Element Distortion**  
 $\Phi_{\text{mod}} = 50,000\Phi$

**Kalman Filter**

Observed Quantities			State Vector Elements	
Epoch	dPhi	dVelocity	dPhi	dVelocity
seconds	radians	meters/second	radians	meters/second
2	3.000e-08	9.133e-02	3.000e-08	9.133e-02
4	6.000e-08	9.183e-02	6.000e-08	9.183e-02
6	8.000e-08	9.204e-02	7.817e-08	9.204e-02
8	1.100e-07	9.244e-02	1.074e-07	9.244e-02
10	1.300e-07	9.193e-02	1.274e-07	9.193e-02
12	1.500e-07	9.274e-02	1.477e-07	9.274e-02
14	1.700e-07	9.374e-02	1.712e-07	9.374e-02
16	2.000e-07	9.425e-02	2.005e-07	9.425e-02
18	2.300e-07	9.344e-02	2.301e-07	9.344e-02
20	2.500e-07	9.324e-02	2.500e-07	9.324e-02
22	2.700e-07	9.214e-02	2.700e-07	9.214e-02
24	2.900e-07	9.344e-02	2.900e-07	9.344e-02
26	3.300e-07	9.445e-02	3.300e-07	9.445e-02
150	2.260e-06	8.601e-02	2.260e-06	8.601e-02
152	2.310e-06	8.560e-02	2.310e-06	8.560e-02
154	2.350e-06	8.540e-02	2.346e-06	8.540e-02

**2-Stage Filter**

Observed Quantities			State Vector Elements	
Epoch	dPhi	dVelocity	dPhi	dVelocity
seconds	radians	meters/second	radians	meters/second
2	3.000e-08	9.133e-02	3.000e-08	9.133e-02
4	6.000e-08	9.183e-02	6.000e-08	9.183e-02
6	8.000e-08	9.204e-02	8.000e-08	9.204e-02
8	1.100e-07	9.244e-02	-1.655e-05	9.244e-02
10	1.300e-07	9.193e-02	-2.172e-05	-1.615e+01
12	1.500e-07	9.274e-02	1.500e-07	9.274e-02
14	1.700e-07	9.374e-02	-5.968e-03	9.374e-02
16	2.000e-07	9.425e-02	5.240e+01	9.425e-02
18	2.300e-07	9.344e-02	1.677e-04	1.405e+03
20	2.500e-07	9.324e-02	2.500e-07	3.974e+08
22	2.700e-07	9.214e-02	2.700e-07	-2.400e+00
24	2.900e-07	9.344e-02	-3.770e+05	9.344e-02
26	3.300e-07	9.445e-02	3.300e-07	2.194e-01
150	2.260e-06	8.601e-02	2.260e-06	-1.593e+18
152	2.310e-06	8.560e-02	-2.939e+15	8.560e-02
154	2.350e-06	8.540e-02	2.350e-06	8.540e-02

#### **4.4 Increase the Magnitude of $\Sigma_1$ .**

**Hypothesis:** When the elements of the observation covariance matrix are increased in magnitude, the filter more sensitive to the error will compute state vectors which diverge from the observed and unmodified calculated values. The filter which is more robust will continue to yield the observed quantities in the first two elements of its state vector.

**Procedure:** Error was introduced by increasing the diagonal elements of  $\Sigma_1$  to 1.0, from  $2.0 \times 10^{-6}$  and  $2.25 \times 10^{-6}$ , respectively, which approximates the diagonal elements of the unmodified transition matrices,  $\Phi_x$  and  $\Phi_y$ . The results were observed to determine which filter was more sensitive.

**Results:** The results at selected epochs are displayed in **Table 5**. Two lines are included for each epoch. The first line is the calculated state vector using the unmodified  $\Sigma_1$ ; the second line is the state vector using the modified matrix,  $\Sigma_{mod}$ . The Kalman filter yielded results which were reduced in magnitude from the observed elements. The 2-Stage Filter, on the other hand, was insensitive to the error in the first two elements, and much less sensitive in the third and fourth elements, as well.

**Analysis:** Consider equation (2-12) which defines the 2-Stage gain matrix:

$$\bar{\bar{K}}_1 = (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T [A_1 (\Theta_1 + \Phi_0 \Sigma_0^0 \Phi_0^T) A_1^T]^{-1} A_1 (A_1^T \Sigma_1^{-1} A_1)^{-1} A_1^T \Sigma_1^{-1} \quad (4-8)$$

During the first epoch,  $\Sigma_1$  only appears at the tail end of the expression. Focusing our attention on the portion of the equation with  $\Sigma_1$  we find:

$$(\mathbf{A}_1^T \Sigma_1^{-1} \mathbf{A}_1)^{-1} \mathbf{A}_1^T \Sigma_1^{-1} \quad (4-9)$$

Substituting the expression  $\epsilon \mathbf{I}$  for  $\Sigma_1^{-1}$ , where  $\epsilon$  is a real number, we see:

$$(\mathbf{A}_1^T \epsilon \mathbf{I} \mathbf{A}_1)^{-1} \mathbf{A}_1^T \epsilon \mathbf{I} \quad (4-9a)$$

which equals:

$$(\epsilon \mathbf{A}_1^T \mathbf{A}_1)^{-1} \mathbf{A}_1^T \epsilon \quad (4-9b)$$

When  $\epsilon$  is pulled out of the g-inverse:

$$\frac{1}{\epsilon} (\mathbf{A}_1^T \mathbf{A}_1)^{-1} \mathbf{A}_1^T \epsilon = (\mathbf{A}_1^T \mathbf{A}_1)^{-1} \mathbf{A}_1^T = \mathbf{A}_1^T \quad (4-9c)$$

it becomes clear that the 2-Stage gain matrix is initially unaffected by the systematic error introduced through the observation covariance matrix. After the initial epoch, however, the gain matrix is not completely immune to systematic error from  $\Sigma_1$  because the updated  $\Sigma_1^0$ , also depends on  $\Sigma_1$ . The result is the subsequent state vector covariance matrices have elements of larger magnitude (meaning less precise information is known about prior state vectors), which affects the lower block of the gain matrix. Consequently, the third and fourth state vector elements are not completely immune to systematic error introduced through the observation covariance matrix.

In this case, we've assumed that our confidence in our observations is low, yet we see that the observable state vector elements pass through the 2-Stage filter, but not the Kalman. This would appear to suggest that the 2-Stage filter is insensitive to

this source of error, however, that conclusion is probably inaccurate. We used the same data set which was obtained with the "good" covariance matrix, and allowed our confidence in the observations erode when we increased the values of the observation covariance matrix to increase. Realistically, our state vector observable elements with such a matrix, even though they would be passed through the 2-Stage filter, could be no more reliable than the quantities which are actually observed given the observation covariance matrix. If those observations are "bad", the corresponding updated state vector elements will not be any better after being processed by the 2-Stage filter. Even though the Kalman filter is adversely affected by this type of error where the 2-Stage filter is not, it cannot be immediately concluded that the results from the 2-Stage filter are "better."



**Table 5**  
**X-Channel State Vector Comparison**  
 $\Sigma_1$  versus  $\Sigma_{\text{mod}} = I_2$

**Kalman Filter**

	Epoch	dLambda	dVelocity	Attitude Angle	Drift
	seconds	radians	meters/sec	radians	radians/sec
$\Sigma_1$	2	-2.443e-05	7.811e-02	1.110e-08	-2.866e-13
$\Sigma_{\text{mod}}$	2	-6.860e-09	2.103e-05	2.987e-12	-7.716e-17
$\Sigma_1$	4	-2.466e-05	7.916e-02	3.297e-08	-1.140e-09
$\Sigma_{\text{mod}}$	4	-1.713e-08	5.536e-05	-1.472e-11	-1.457e-11
$\Sigma_1$	6	-2.471e-05	7.977e-02	4.576e-08	-3.801e-09
$\Sigma_{\text{mod}}$	6	-3.081e-08	1.032e-04	-2.851e-10	-8.795e-11
$\Sigma_1$	8	-2.476e-05	8.037e-02	3.341e-08	-9.747e-09
$\Sigma_{\text{mod}}$	8	-4.789e-08	1.648e-04	-1.487e-09	-2.948e-10
$\Sigma_1$	10	-2.480e-05	8.038e-02	3.819e-08	-9.898e-09
$\Sigma_{\text{mod}}$	10	-6.835e-08	2.398e-04	-4.972e-09	-7.374e-10

**2-Stage Filter**

	Epoch	dLambda	dVelocity	Attitude Angle	Drift
	seconds	radians	meters/sec	radians	radians/sec
$\Sigma_1$	2	-2.461e-05	7.877e-02	1.119e-08	-2.891e-13
$\Sigma_{\text{mod}}$	2	-2.461e-05	7.877e-02	1.119e-08	-2.891e-13
$\Sigma_1$	4	-2.466e-05	7.917e-02	3.490e-08	-4.382e-10
$\Sigma_{\text{mod}}$	4	-2.466e-05	7.917e-02	3.610e-08	-3.646e-13
$\Sigma_1$	6	-2.471e-05	7.978e-02	4.919e-08	-3.075e-09
$\Sigma_{\text{mod}}$	6	-2.471e-05	7.978e-02	6.142e-08	-9.228e-13
$\Sigma_1$	8	-2.476e-05	8.038e-02	3.830e-08	-9.020e-09
$\Sigma_{\text{mod}}$	8	-2.476e-05	8.038e-02	8.711e-08	-2.483e-12
$\Sigma_1$	10	-2.480e-05	8.038e-02	4.540e-08	-9.034e-09
$\Sigma_{\text{mod}}$	10	-2.480e-05	8.038e-02	1.123e-07	-2.494e-12

## CHAPTER V

### CONCLUSION

#### 5.1 Summary of Results

During the process of analyzing the Kalman and 2-Stage filters, attention was focused on two of the covariance matrices,  $\Theta_1$  and  $\Sigma_1$ , in addition to the transition matrices,  $\Phi_x$  and  $\Phi_y$ . When system error was introduced into the state transition covariance matrix,  $\Theta_1$ , by making it large so as to approach  $\infty$ , the 2-Stage Filter was unaffected in its observed state vector elements, but it was unstable in the elements which were not observable. The Kalman Filter produced state vectors which were identical to the 2-Stage Filter. When system error was introduced via the transition matrices,  $\Phi_x$  and  $\Phi_y$ , the two filters appeared to generate state vectors of statistically insignificant differences. When system error was introduced by distorting the transition matrices sufficiently to distort the observed elements, the 2-Stage Filter produced results which exceeded the precision of the Kalman Filter, however, there appears to exist an upper limit at which point the 2-Stage results are adversely affected by system rounding. When error was introduced via the observation equation covariance matrix,  $\Sigma_1$ , the Kalman Filter yielded distorted state vector element values, whereas the 2-Stage Filter was unaffected regarding its observed state vector elements.

## **5.2 Conclusions, and Areas of Future Research**

The 2-Stage Filter can only be considered to be more robust than the Kalman Filter if the source of the model error is in the observations. In this case, the 2-Stage Filter is stable in the observed state vector elements, where the Kalman Filter is unstable in all of its state vector elements. All other sources of error, i.e. state transition covariance matrix, and the transition matrices, the two filters generated results which were of no significant difference.

In light of the fact that the 2-Stage Filter is definitely the less efficient of the two filters (it requires at least 30% more lines of code to perform its computations) there appears to be little reason to expend much energy investigating it further. However, more detailed analysis may be useful in order to investigate the reasons for the apparent limit at which the 2-Stage Filter is affected by system rounding. Additionally, a closer look at the effects of the error on the state vector covariance matrix,  $\Sigma_0^0$ , may provide insight into the differences between the Kalman and 2-Stage filters.

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